Journal of Approximation Theory **116**, 416–424 (2002)
doi:10.1006/jath.2002.3678

# NOTE

# A Note on a Remez-Type Inequality for Trigonometric Polynomials

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*Communicated by Tamás Erdélyi*

Received February 7, 2001; accepted in revised form December 31, 2001

We obtain sharp bounds, in the uniform norm along the unit circle $\mathbb{T}$, of exponentials of logarithmic potentials, if the logarithmic capacity of the subset of $\mathbb{T}$, where they are at most 1, is known. 

*Key Words:* Remez inequality; polynomial; exponential of a potential; logarithmic capacity.

## 1. INTRODUCTION

Let $|A|$ be the linear measure (length) of a set $A$ in the complex plane $\mathbb{C}$. By $\mathbb{P}_n$ we denote the set of all complex polynomials of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$. Let

$$\Pi(p) := \{z \in \mathbb{C} : |p(z)| > 1\}, \qquad p \in \mathbb{P}_n.$$

From the numerous generalizations of the classical Remez inequality (see, for example, [3, 5]), we cite one result which is a direct consequence of the trigonometric version of the Remez inequality (and is equivalent to this trigonometric version, up to constants).

Assume that $p \in \mathbb{P}_n$, $\mathbb{T} := \{z: |z| = 1\}$ and

$$|\mathbb{T} \cap \Pi(p)| \leqslant s, \qquad 0 < s \leqslant \frac{\pi}{2}. \tag{1.1}$$



0021-9045/02 $35.00

Then, $q(t) := |p(e^{it})|^2$ is a trigonometric polynomial of degree at most $n$ and, by the Remez-type inequality on the size of trigonometric polynomials (cf. [4, Theorem 2; 3, p. 230]), we have

$$\|p\|_{\mathbb{T}} \leqslant e^{2sn}, \qquad 0 < s \leqslant \frac{\pi}{2}. \tag{1.2}$$

Here, $\|\cdot\|_A$ means the uniform norm along $A \subset \mathbb{C}$.

Our aim is to prove an analogue of (1.1)–(1.2) in which we use logarithmic capacity instead of the length. Our main result deals not only with polynomials, but also with exponentials of potentials (see [5]).

We refer to the basic notions of potential theory (such as capacity, potential, Green's function, equilibrium measure) without special citations. All these notions and their properties can be found in [10, 11].

Given a nonnegative Borel measure $\nu$ with compact support in $\mathbb{C}$ and finite total mass $\nu(\mathbb{C}) > 0$, as well as a real number $c \in \mathbb{R}$, we say that

$$Q_{\nu, c}(z) := \exp(c - U^{\nu}(z)), \qquad z \in \mathbb{C},$$

where

$$U^{\nu}(z) := \int \log \frac{1}{|\zeta - z|}\, d\nu(\zeta), \qquad z \in \mathbb{C},$$

is the logarithmic potential of $\nu$, is an exponential of a potential of degree $\nu(\mathbb{C})$.

Let

$$E_{\nu, c} := \{z \in \mathbb{T} : Q_{\nu, c}(z) \leqslant 1\}.$$

Our main result can be formulated as follows. Denote by cap $K$ the logarithmic capacity of a set $K \subset \mathbb{C}$.

THEOREM. *Let* $0 < \delta < 1$. *Then, the condition*

$$\operatorname{cap} E_{\nu, c} \geqslant \delta$$

*implies that*

$$\|Q_{\nu, c}\|_{\mathbb{T}} \leqslant \left(\frac{1 + \sqrt{1 - \delta^2}}{\delta}\right)^{\nu(\mathbb{C})}.$$

The analogue of this theorem, with the unit interval $[-1, 1]$ instead of $\mathbb{T}$, is proved in [1].

*Remark* 1. In order to examine the sharpness of this theorem we consider the following example. For any compact set $E \subset \mathbb{C}$ with $\operatorname{cap} E > 0$, denote by

$$g(z, \infty, \overline{\mathbb{C}} \setminus E), \qquad z \in \overline{\mathbb{C}} \setminus E,$$

(where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane) the Green's function of $\overline{\mathbb{C}} \setminus E$ with pole at infinity.

Let $0 < \alpha < \pi/2$, and let $L = L_\infty := \{e^{i\theta} : 2\alpha \leqslant \theta \leqslant 2\pi - 2\alpha\}$. Since the function

$$z = \Psi(w) = -w \frac{w-a}{1-aw},$$

where $a = 1/\cos\alpha$, maps $\Delta := \{w\colon |w| > 1\}$ onto $\Omega := \overline{\mathbb{C}} \setminus L$ (cf. [6]) and since the Green's function of $\Omega$ with pole at $\infty$ can be defined via the inverse function $\Phi := \Psi^{-1}$ by the formula

$$g(z, \infty, \Omega) = \log |\Phi(z)|, \qquad z \in \Omega,$$

we have

$$\operatorname{cap} L = \lim_{w \to \infty} \frac{\Psi(w)}{w} = \frac{1}{a} = \cos\alpha \tag{1.3}$$

as well as

$$\begin{aligned} \max_{z \in \mathbb{T} \setminus L} g(z, \infty, \Omega) &= g(1, \infty, \Omega) = \log |\Phi(1)| \\ &= \log(a + \sqrt{a^2 - 1}\,) = \log \frac{1 + \sqrt{1 - (\operatorname{cap} L)^2}}{\operatorname{cap} L}. \end{aligned} \tag{1.4}$$

Let $c = c_\alpha := -\log \operatorname{cap} L$ and let $\nu = \nu_\alpha$ be the equilibrium measure for $L$; that is, $\nu(\mathbb{C}) = 1$. Since for $z \in \mathbb{C}$,

$$U^\nu(z) = -g(z, \infty, \overline{\mathbb{C}} \setminus L) - \log \operatorname{cap} L,$$

and therefore

$$Q_{\nu, c}(z) = \exp(g(z, \infty, \overline{\mathbb{C}} \setminus L)),$$

we have $E_{\nu, c} = L$ as well as

$$\|Q_{\nu, c}\|_{\mathbb{T}} = \frac{1 + \sqrt{1 - (\operatorname{cap} L)^2}}{\operatorname{cap} L}.$$

This shows the exactness of the theorem.

*Remark* 2. Let $p(z)=c\prod_{j=1}^{n}(z-z_j)$, $c=(\text{const})\neq 0$, be a complex polynomial of degree $n$, and let

$$\nu_n := \sum_{j=1}^{n} \delta_{z_j},$$

where $\delta_z$ is the Dirac unit measure in a point $z\in\mathbb{C}$. For $z\in\mathbb{C}$, we have

$$Q_{\nu_n,\log|c|} = \exp\left(\log|c| + \log\prod_{j=1}^{n}|z-z_j|\right) = |p(z)|.$$

Therefore, applying the above theorem, we obtain the following, for $0<\delta<1$: For $p\in\mathbb{P}_n$ the condition

$$\operatorname{cap}\mathbb{T}\setminus\Pi(p)\geqslant\delta \tag{1.5}$$

implies

$$\|p\|_{\mathbb{T}}\leqslant\left(\frac{1+\sqrt{1-\delta^2}}{\delta}\right)^n. \tag{1.6}$$

In our approach, we exploit the following simple connection between estimates which expresses the possible growth of a polynomial with a known norm on a given compact set $E\subset\mathbb{C}$ and the behavior of the Green's function for $\mathbb{C}\setminus E$.

Let the logarithmic capacity of $E$ be positive and let $\Omega:=\mathbb{C}\setminus E$ be connected. For $z\in\Omega$ and $u>0$, the following two conditions are equivalent:

(i) $g(z,\infty,\Omega)\leqslant u$;

(ii) for any $p\in\mathbb{P}_n$, and $n\in\mathbb{N}$,

$$|p(z)|\leqslant e^{un}\|p\|_E.$$

Indeed, (i) $\Rightarrow$ (ii) follows from the Bernstein– Walsh lemma and (ii) $\Rightarrow$ (i) is a simple consequence of a result by Myrberg and Leja (see [9, p. 333]).

We study the properties of the Green's function by methods of geometric function theory (involving symmetrization) which allows, according to the implication (i) $\Rightarrow$ (ii), us to obtain results similar to (1.5)–(1.6).

Note that the sharpness of results for the Green's function, by virtue of the equivalence of (i) and (ii), implies the (asymptotic) sharpness of the corresponding Remez-type inequalities for polynomials (1.5)–(1.6).

*Remark* 3. Since for any $E\subset\mathbb{T}$ we have $\operatorname{cap}E\geqslant\sin\frac{|E|}{4}$ (see [10]), (1.5)–(1.6) imply the following refinement of (1.1)–(1.2): For $p\in\mathbb{P}_n$ the condition

$$|\mathbb{T}\cap\Pi(p)|\leqslant s,\qquad 0<s<2\pi, \tag{1.7}$$

implies

$$\|p\|_{\mathbb{T}} \leqslant \left( \frac{1+\sin \frac{s}{4}}{\cos \frac{s}{4}} \right)^n. \tag{1.8}$$

This result is also sharp in the following sense. Let $0 < s < 2\pi$, $\alpha = s/4$, and let $L = L_\alpha$ be defined as in Remark 1. By (1.3) and (1.4),

$$g(1, \infty, \Omega) = \log \frac{1+\sin \frac{s}{4}}{\cos \frac{s}{4}}.$$

We denote by $f_n(z)$ the Fekete polynomial for a compact set $L$ (see [11]). Hence, condition (1.7) holds for the polynomial $p(z) = p_n(z) := f_n(z)/\|f_n\|_L$. At the same time, since

$$\lim_{n \to \infty} \left( \frac{|f_n(z)|}{\|f_n\|_L} \right)^{1/n} = \exp(g(z, \infty, \Omega)), \qquad z \in \Omega \setminus \{\infty\}$$

(see [11, p. 151]), we have

$$\lim_{n \to \infty} |p(1)|^{1/n} = \frac{1+\sin \frac{s}{4}}{\cos \frac{s}{4}}.$$

The theorem is a straightforward consequence of its following particular case.

Lemma. *Let $E \subset \mathbb{T}$ be a compact set with $0 < \operatorname{cap} E < 1$. Then,*

$$\sup_{z \in \mathbb{T} \setminus E} g(z, \infty, \mathbb{C} \setminus E) \leqslant \log \left( \frac{1+\sqrt{1-(\operatorname{cap} E)^2}}{\operatorname{cap} E} \right).$$

## 2. PROOFS

The proof of the lemma is based on utilizing the circular symmetrization which is defined as follows.

For $r > 0$, let $C(r) := \{\zeta: |\zeta| = r\}$. For some domain $H \in \mathbb{C}$, we denote by $H^*$ the circular symmetrization of $H$ with respect to 0, which is defined as follows: for $t \geqslant 0$, set

$$H_t := \{\theta \in [0, 2\pi] : te^{i\theta} \in H\},$$

and

$$h_t := \begin{cases} C(t), & H_t = [0, 2\pi], \\ \{te^{i\phi}: |\phi| < |H_t|/2\}, & [0, 2\pi] \setminus H_t \neq \varnothing. \end{cases}$$

Then,

$$H^* := \bigcup_{t \geqslant 0} h_t.$$

In our application $H$ is a simply connected domain in the unit disk $\mathbb{D} := \{z: |z| < 1\}$ and $0 \in H$. Let $w = h(z)$ be a conformal mapping of $\mathbb{D}$ onto $H$ such that $h(0) = 0$. The quantity $r_0(H) := |h'(0)|$ is called the inner radius of $H$ at 0. It is well known that the inner radius increases with expanding domains and symmetrization (see [7]). It means that $r_0(H) \leqslant r_0(H^*) \leqslant r_0(\tilde{H})$, where

$$\tilde{H} := \mathbb{D} \setminus [-1, -m], \qquad m := \min_{w \in \partial H} |w|.$$

The inner radius of $\tilde{H}$ can be simply computed by using the Joukowski mapping

$$z = J(w) = \frac{1}{2}\left(w + \frac{1}{w}\right);$$

that is,

$$r_0(\tilde{H}) = \frac{4}{2 + m + \frac{1}{m}}. \tag{2.1}$$

*Proof of the lemma.* Let $\mu_E$ be the equilibrium measure of a compact set $E \subset \mathbb{T}$ with cap $E > 0$. According to [10, p. 210], the function

$$h(z) = z(\operatorname{cap} E)^2 \exp\left[-2\int \log(1 - \bar{\zeta} z)\, d\mu_E(\zeta)\right], \qquad z \in \mathbb{D},$$

maps $\mathbb{D}$ conformally onto a starlike domain $H = h(\mathbb{D}) \subset \mathbb{D}$. Since

$$U^{\mu_E}(z) = -g(z, \infty, \mathbb{C} \setminus E) - \log \operatorname{cap} E, \qquad z \in \mathbb{C} \setminus E,$$

we have for $z \in \mathbb{D}\setminus\{0\}$,

$$\log\left|\frac{h(z)}{z}\right| = 2\log \operatorname{cap} E + 2U^{\mu_E}(z) = -2g(z, \infty, \mathbb{C}\setminus E). \tag{2.2}$$

In particular, taking the limit as $z \to 0$, we obtain

$$-\tfrac{1}{2}\log|h'(0)| = g(0, \infty, \mathbb{C}\setminus E) = -\log \operatorname{cap} E.$$

Therefore,

$$\operatorname{cap} E = r_0(H)^{1/2}. \tag{2.3}$$

Let $b := \sup_{z\in\mathbb{T}} g(z, \infty, \mathbb{C}\setminus E)$. By (2.2)

$$\inf_{w\in\partial H} |w| = e^{-2b}.$$

At the same time, by (2.1) and (2.3) we obtain

$$(\operatorname{cap} E)^2 = r_0(H) \leqslant r_0(\tilde{H}) = \frac{4}{2+e^{2b}+e^{-2b}},$$

which yields

$$b \leqslant \log\frac{1+\sqrt{1-(\operatorname{cap} E)^2}}{\operatorname{cap} E}.$$

∎

*Proof of the theorem.* Let $E \subset E_{\nu,c}$ be a compact set. The general version of the Bernstein–Walsh inequality (cf. [8, p. 532]) and the lemma above yield, for any $z \in \mathbb{T}\setminus E_{\nu,c}$,

$$Q_{\nu,c}(z) \leqslant \exp(\nu(\mathbb{C})\, g(z, \infty, \mathbb{C}\setminus E))\, \|Q_{\nu,c}\|_E \leqslant \left(\frac{1+\sqrt{1-(\operatorname{cap} E)^2}}{\operatorname{cap} E}\right)^{\nu(\mathbb{C})}.$$

Since $E$ is arbitrary, we obtain

$$Q_{\nu,c}(z) \leqslant \left(\frac{1+\sqrt{1-(\operatorname{cap} E_{\nu,c})^2}}{\operatorname{cap} E_{\nu,c}}\right)^{\nu(\mathbb{C})}.$$

∎

*Remark* 4. Note that the Remez-type inequality (1.7)– (1.8) (more precisely its analog concerning Green's function) has another simple proof via symmetrization. Namely, let $E \subset \mathbb{T}$ be a compact set such that $|\mathbb{T} \setminus E| = s \in (0, 2\pi)$. Set

$$E^* := \left\{ e^{i\theta}: \frac{s}{2} \leqslant \theta \leqslant 2\pi - \frac{s}{2} \right\}.$$

It follows from a result by Bernstein (see [2, p. 142]) that for $r > 0$ and $z \in \mathbb{T} \setminus E$,

$$\sup_\theta g(z, re^{i\theta}, \mathbb{C} \setminus E) \leqslant \sup_\theta g(1, re^{i\theta}, \mathbb{C} \setminus E^*).$$

Therefore, taking the limit as $r \to \infty$ and using (1.3) and (1.4), we obtain, for $z \in \mathbb{T} \setminus E$,

$$g(z, \infty, \mathbb{C} \setminus E) \leqslant g(1, \infty, \mathbb{C} \setminus E^*) = \log \frac{1 + \sin \frac{s}{4}}{\cos \frac{s}{4}}.$$

Hence, using the Bernstein–Walsh lemma in a standard way, we get (1.7)–(1.8).

## ACKNOWLEDGMENT

The author is indebted to R. Varga for his helpful comments.

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